Indian Statistical Institute, Bangalore Centre
B.Math (Hons) II Year, Second Semester
Semestral Examination
Algebra IV

Time: 10.00AM - 1.00PM May 13, 2013 Instructor: Bhaskar Bagchi

Full Marks : 100.

1. Recall that an algebraic closure $\bar{F}$ of a field F is an algebraic extension of F which is algebraically closed.
   (a) Show that $\bar{F}$ is an algebraic closure of F iff $\bar{F}$ is a maximal element in the class of all algebraic extensions of F, partially ordered by inclusion.
   (b) If $\sigma : E_1 \to E_2$ is an F-homomorphism between two algebraic extensions of F, then show that σ extends to an F-homomorphism $\bar{\sigma} : \bar{F_1} \to \bar{F_2}$ between two algebraic closures of F.

   [5+15=20]

2. (a) Let F be a finite field of order q. Let E be an extension of F of degree n. Then show that every irreducible polynomial of degree n over F splits completely over E.
   (b) Prove that the product of all the monic irreducible polynomials over F of degree dividing n equals $X^{q^n} - X$.
   (c) Prove that the total number of monic irreducible polynomials of degree n over F is $\frac{1}{n} \sum\limits_{d/n} \mu\left(\frac{n}{d}\right) q^d$, where μ is the Möbius function.

   [5+10+15=30]

3. (a) State completely and precisely the fundamental theorem of Galois theory.
   (b) If $F \subseteq E$ is a Galois extension of finite degree with Galois group $G, B_1 \subseteq B_2$ are two intermediate fields with $H_1 \supseteq H_2$ the corresponding subgroups of G under the Galois correspondence, then show that B_2 is Galois over B_1 iff H_2 is normal in H_1.

   [5+15=20]

4. (a) If X is a division ring (skew field) and $F = \{x \in X : xy = yx \ \forall y \in X\}$ then show that F is a field.
   (b) If X is finite, F is of order q, and X is n-dimensional over F, then show that there are m proper divisors $a_1, \cdots, a_m$ of n such that $q^n - 1 = q - 1 = \sum\limits_{i=1}^{m} \frac{q^n-1}{q^{ai}-1}$, where m is the number of non-trivial conjugacy classes of X^*.
   (c) Show that we must have $m = 0$.

   [5+5+10=20]

5. Prove that, for $n \geq 3$, a regular n-gon is constructible by Euclidean methods iff $\varphi(n)$ is a power of two. [10]